QCD running coupling constant
in the timelike region

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Abstract

By using a non-perturbative expansion and the dispersion relation for the Adler D–function we propose a new method for constructing the QCD effective coupling constant in the timelike region.

The perturbative expansion is a powerful tool for performing calculations in quantum chromodynamics. Perturbation theory with renormalization group improvement has been widely applied to the description of various processes in both the spacelike and timelike regions. However, as is well known, the running coupling constant $\alpha_S(Q^2)$, the expansion parameter in QCD perturbation theory, is usually defined in the Euclidean range of momentum. If one wishes to know the "running coupling constant" in the timelike region, and how it evolves, one must make use of some analytic continuation from the Euclidean (spacelike) region to the timelike region. To this end one usually applies the dispersion relation for the Adler D–function [1]

$$D(q^2) \; = \; - \, q^2 \int_0^\infty \frac{d\,s}{(s - q^2)^2} \, R(s) \; , \tag{1}$$

where

$$D(q^2) \; = \; q^2 \, (-\frac{d}{d\,q^2}) \, \Pi(q^2) \tag{2}$$

and the function $\Pi(q^2)$ is defined by

$$\mathrm{i} \int d^4x \, \exp(\mathrm{i}\,q.x) \, \langle 0| \, T \, \{J_\mu(x) \, J_\nu(0)\}|0\rangle \; \propto \; (q_\mu \, q_\nu \, - \, g_{\mu\nu} \, q^2) \, \Pi(q^2) \tag{3}$$

and

$$\begin{aligned}
R(s) \;\; &= \;\; \frac{1}{2\pi\mathrm{i}} \, [\, \Pi(s + \mathrm{i}\,\epsilon) \, - \, \Pi(s - \mathrm{i}\,\epsilon) \,] \; = \; \frac{1}{\pi} \, \mathrm{Im} \, \Pi(s) \\
&= \;\; -\frac{1}{2\pi\mathrm{i}} \int_{s-\mathrm{i}\epsilon}^{s+\mathrm{i}\epsilon} \frac{d\,z}{z} \, D(z) \, .
\end{aligned} \tag{4}$$

The assumptions for writing these equations are the following: the function $\Pi(q^2)$ is an analytic function in the complex q^2–plane with a cut along the positive real axis, and the dispersion relation for the D–function does not require any subtractions. The contour in Eq. (4) goes from the point $z \, = \, s - \mathrm{i}\,\epsilon$ to the point $z \, = \, s + \mathrm{i}\,\epsilon$ and lies in the region where the function $D(z)$ is an analytic function of z (see Figure 1).

Let us represent the functions $D(q^2)$ and $R(q^2)$ in the form

$$\begin{aligned}
D(q^2) \;\; &\propto \;\; \sum_f Q_f^2 \, [\, 1 \, + \, d_0 \, \lambda_t(q^2) \, + \, d_1 \, \lambda_t^2(q^2) \, + \cdots] \\
&\equiv \;\; \sum_f Q_f^2 \, [\, 1 \, + \, d_0 \, \lambda_t^{\mathrm{eff}}(q^2) \,] \, ,
\end{aligned} \tag{5}$$

and

$$R(s) \quad \propto \quad \sum_f Q_f^2 \, [\, 1 \, + \, r_0 \, \lambda_s(s) \, + \, r_1 \, \lambda_s^2(s) \, + \cdots]$$

$$\equiv \quad \sum_f Q_f^2 \, [\, 1 \, + \, r_0 \, \lambda_s^{\text{eff}}(s) \,], \tag{6}$$

where we use the definition $\lambda = \alpha_S/(4\pi)$. The index t means "t–channel" (the spacelike region) and s means "s–channel" (the timelike region). The coefficients d_0 in Eq. (5) and r_0 in Eq. (6) are numerically equal; thus Eq. (1) leads to a dispersion relation relating the effective coupling constants in the t and s channels:

$$\lambda_t^{\text{eff}}(q^2) \; = \; -\, q^2 \int_0^\infty \frac{d\,s}{(s - q^2)^2} \, \lambda_s^{\text{eff}}(s) \; . \tag{7}$$

The solution of Eq. (7), given the analytic properties of $\lambda_t^{\text{eff}}(t)$, is

$$\lambda_s^{\text{eff}}(s) \; = \; -\, \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\epsilon}^{s+\mathrm{i}\epsilon} \frac{d\,z}{z} \, \lambda_t^{\text{eff}}(z) \tag{8}$$

Thus, Eq. (8) serves to define the running coupling constant in the timelike region. If we use the leading one–loop approximation for the running coupling constant λ_t^{eff} in the t–channel,

$$\lambda_t^{\text{eff}}(z) \; = \; \frac{1}{b_0 \, \ln(-z/\Lambda^2),} \tag{9}$$

where $b_0 \, = \, 11 - 2n_f/3$ is the first coefficient of the β–function and n_f is the number of flavours, we obtain

$$b_0 \, \lambda_s^{\text{eff}}(s) \; = \; \frac{1}{\pi} \arctan \frac{\pi}{\ln(s/\Lambda^2)} \; = \; \frac{1}{\ln(s/\Lambda^2)} - \frac{1}{3} \frac{\pi^2}{\ln^3(s/\Lambda^2)} + \cdots \tag{10}$$

Thus, one might conclude that the main effect of the analytic continuation from the spacelike to the timelike region is an additional contribution (the so-called π^2–term) of order $1/\ln^3(s/\Lambda^2)$. The π^2–terms play an important role in the analysis of various processes. For example, in the τ–lepton region, where $\alpha_S \simeq 0.35$ (see, for example, a discussion in Ref. [2]), the π^2–correction is about twenty percent and the ratio $R(s) \, = \, \lambda_s(s)/\lambda_t(q^2) \simeq 0.8$ for $s \, = \, -q^2 \, = \, M_\tau^2$. The π^2–terms and the problem of the construction of the running coupling constant in the timelike domain have been discussed in detail in Refs. [3-7].

However, the one–loop approximation to λ_t^{eff} breaks the analytical properties of $\lambda_t^{\text{eff}}(z)$ for which we wrote the solution (8). Indeed, the function $\lambda_t^{1-\text{loop}}$ has an infrared singularity

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at $z = -\Lambda^2$, which contradicts the assumption that the function $\lambda_t^{\mathrm{eff}}(z)$ has only a cut in the complex z-plane. A consequence of this is the fact that Eq. (7) does not reproduce the original one–loop coupling constant in the t–channel if we use for $\lambda_s^{\mathrm{eff}}(s)$ the "solution" in the form of Eq. (8). In order to maintain the analytic properties of λ^{eff} one must make some modification to the initial coupling constant, which is to be substituted into Eq. (8).

From a phenomenological point of view such modifications can be made in an *ad hoc* manner, with no fundamental justification. In the present note, however, we construct the running coupling constant in the timelike domain in the well-motivated framework of the non-perturbative method proposed in Ref. [8], which allows one systematically to determine the low energy structure in QCD. The method is based on the introduction of a new small expansion parameter. For small coupling constant $\alpha_S(Q^2)$ the series thus obtained coincides with standard perturbation theory, so that high energy physics is essentially unaffected. Moreover the method continues to be valid in the nonperturbative region, where $\alpha_S(Q^2) \geq 1$, since the expansion parameter in this approach remains small.

The approach is related to the method of the Gaussian effective potential [9-12], the linear δ–expansion [13,14] and the approach based on variational perturbation theory [15,16]. Within these approaches the investigated quantity is written from the very beginning in the form of a series which provides a well-determined algorithm for calculating corrections up to any order. In addition, the series contains some auxiliary parameters which allow one to control the convergence properties of the series and to construct an expansion based on a new small parameter, as in this paper. The exact value of a given quantity is independent of these parameters. However, the corresponding approximate result contains a residual dependence upon those variables, and we may choose the latter so as to provide the best approximation to the quantity. A similar situation holds in standard perturbation theory for the total cross section for $e^+ e^-$ annihilation into hadrons: the physical quantity $R_{e^+e^-}$ is independent of a renormalization scheme, whereas a finite perturbative approximation to $R_{e^+e^-}$ is scheme dependent (in this connection see Refs. [17,18]).

The method described in Ref. [8] starts with the standard action of QCD, written as

$$S(A, q, \varphi) = S_2(A) + S_2(q) + S_2(\varphi) + g\, S_3(A, q, \varphi) + g^2\, S_4(A) \,, \qquad (11)$$

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where $S_2(A)$, including the gauge–fixing term, $S_2(q)$ and $S_2(\varphi)$ are the standard free actions of the gluon, quark and ghost fields, while $S_3(A, q, \varphi)$ contains the trilinear vertices and $S_4(A)$ the four-point gluon vertex. By use of an auxiliary field $\chi_{\mu\nu}$ the latter action can be rewritten as a trilinear action between A and χ, so that S becomes

$$S(A, q, \varphi, \chi) = S_2(q) + S_2(\varphi) + S_2(\chi) + S(A, \chi) + g\, S_3(A, q, \varphi). \tag{12}$$

We then introduce a new split between the free and interaction parts of the action according to

$$S(A, q, \varphi, \chi) = S_0'(A, q, \varphi, \chi) + S_I'(A, q, \varphi, \chi), \tag{13}$$

where

$$S_0' = \zeta^{-1}[S(A, \chi) + S_2(q) + S_2(\varphi)] + \xi^{-1} S_2(\chi), \tag{14}$$

and

$$S_I' = g\, S_3(A, q, \varphi) - (\zeta^{-1} - 1)\,[\, S(A, \chi) + S_2(q) + S_2(\varphi)] - (\xi^{-1} - 1)\, S_2(\chi), \tag{15}$$

thus introducing two variational parameters ξ, ζ. However, it turns out [8] that the two parameters must be related by $\xi = \zeta^3$ in order to preserve gauge invariance. After a rescaling of the fields we obtain the following expression for a general Green function:

$$G(\cdots) = \int D_{QCD}\,(\cdots)\, V(A, q, \varphi)\, \exp(i\, S_0), \tag{16}$$

with

$$\begin{aligned}
V(A, q, \varphi) &= \sum_n \sum_{k=0}^n \frac{1}{(n-k)!}\Big(-\frac{\partial}{\partial\kappa}\Big)^{n-k} \frac{i^k}{k!} \frac{1}{[\,1 + \kappa\,(\zeta^{-1} - 1)\,]^{\nu/2}} \\
&\quad \times\ [g_3\, S_3(A, q, \varphi)]^k \exp\{\, i\,[\, g_4^2\, S_4(A)\,]\,\},
\end{aligned} \tag{17}$$

where

$$g_3 = \frac{g}{[\,1 + \kappa\,(\zeta^{-1} - 1)\,]^{3/2}}\,, \qquad g_4 = \frac{g}{[\,1 + \kappa\,(\xi^{-1} - 1)\,]^{1/2}} \tag{18}$$

Here S_0 is the standard free action of QCD and κ is a parameter introduced for convenience which is set equal to 1 at the end of the calculation.

Analysis of the structure of this variational perturbation series shows that it can be organized in powers of the new small parameter $a \equiv 1 - \zeta$ if the standard coupling constant g is related to a by

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{1}{C}\frac{a^2}{(1-a)^3}, \tag{19}$$

where C is a positive constant. It is clear that for all values of $\lambda \geq 0$ the expansion parameter a obeys the inequality $0 \leq a < 1$. The variational parameter C is fixed, for example, by requiring the correct properties of the β–function in the infrared region, as is discussed in more detail below.

The result of the expansion for the expression (17) to order a^3 is

$$
\begin{aligned}
V &= 1 + a\, A_3 + a^2 \Big[\frac{1}{2} A_3{}^2 + A_4 + \frac{3}{2} A_3 \Big] \\
&+ a^3 \Big[\frac{1}{6} A_3{}^3 + \frac{3}{2} A_3{}^2 + A_3 A_4 + 3 A_4 + \frac{15}{8} A_3 \Big] + O(a^4),
\end{aligned}
\tag{20}
$$

where $A_3 = (\mathrm{i}S_3)/P$, $A_4 = (\mathrm{i}S_4)/P^2$ and $P = C/(4\pi)^2$.

Within this approach the one–loop β–function is

$$
\beta^{(3)}(\lambda) = -\frac{b_0}{C^2} \frac{a^4}{(2+a)\,(1-a)^2}\,(2 + 9\,a).
\tag{21}
$$

Here we must make some comments. The one–loop level that we use here allows us to calculate the renormalization constant Z_λ both to $O(a^2)$ and to $O(a^3)$. The expression (21) gives the $O(a^3)$ result, while $\beta^{(2)}(\lambda)$ is obtained by simply omitting the term $9\,a$ in the final bracket of Eq. (21). We will apply the phenomenon of induced convergence whereby the variational parameter is not fixed for all orders but rather depends on the order of the expansion. As was empirically noticed in Ref. [19] the results seem to converge if one changes the variational parameter from order to order using some variational principle. The induced convergence phenomenon has been discussed in detail in Ref. [20]. In Ref. [14] the convergence of the optimized δ–expansion has been proved in the cases of zero and one dimensions.

The constant C is determined by insisting that at an appropriately large value of λ the β–function should behave like $-\beta(\lambda)/\lambda \simeq 1$, which ensures the correct infrared behaviour of the running coupling constant: $\alpha_S(Q^2) \sim Q^{-2}$. This singular infrared behaviour of the coupling constant leads to the linear part of the quark–antiquark potential which is in agreement with the phenomenology of hadron spectroscopy. In $O(a^2)$ and $O(a^3)$ we find $C^{(2)} = 0.977$ and $C^{(3)} = 4.1$ respectively [21].

By using the renormalization group equation

$$
\ln \frac{Q^2}{Q_0^2} = \int_{\lambda_0}^{\lambda} \frac{d\,\lambda'}{\beta(\lambda')},
\tag{22}
$$

we obtain

$$\ln \frac{Q^2}{Q_0^2} = \frac{C^{(i)}}{2\,b_0}\left[\,f^{(i)}(a) - f^{(i)}(a_0)\,\right],\tag{23}$$

where

$$f^{(2)}(a) = \frac{2}{a^2} + \frac{12}{a} + 21\ln\frac{1-a}{a} - \frac{9}{1-a},$$

$$f^{(3)}(a) = \frac{2}{a^2} - \frac{6}{a} - 48\ln a - \frac{18}{11}\frac{1}{1-a} + \frac{624}{121}\ln\left(1-a\right) + \frac{5184}{121}\ln\left(1 + \frac{9}{2}a\right).\tag{24}$$

The parameter a_0 and the momentum Q_0 in Eq. (23) are defined by some normalization point for our effective coupling constant

$$\lambda_t^{(3)}(q^2) = \frac{a^2}{C^{(3)}}\left(\,1 + 3\,a\,\right).\tag{25}$$

The effective coupling constant $\lambda_t^{(2)}(q^2)$ is obtained from Eq. (25) by omitting the term $3a$ in the final bracket and using the constant $C^{(2)}$ instead of $C^{(3)}$. To fix a_0 we use the value of the running coupling constant measured at the τ–lepton mass [2]: $Q_0 = M_\tau = 1.777$ GeV and $\alpha_S(M_\tau) = 0.35$. In Figure 2 we show the behaviour of our running expansion parameter $a(Q^2)$ in the t–channel, obtained from (23) using $f^{(3)}(a)$. As one can see, this parameter is indeed small in a wide range of Q^2, and approaches 1 for very small Q^2. Moreover, the effective expansion parameter in the method is actually $a^2(Q^2)$, with the $O(a^4)$ coming in at two–loop level. Thus, the real expansion parameter is more or less that shown in Figure 2.

Note that at large $Q^2 = -q^2$ the effective coupling constant (25) essentially coincides with the perturbative coupling constant. However, at small Q^2 their behaviours are quite different. While the perturbative coupling constant has a singularity at $Q^2 = \Lambda^2$, the effective coupling (25) is finite at all values of $Q^2 \geq 0$. As follows from Eq. (23), the effective coupling constant (25) has a cut along the positive q^2–axis. Thus, our approximation for the effective coupling constant in the t–channel does not contradict the dispersion relation (7), and we can make use of Eq. (8) to find the effective coupling constant in the s–channel.

By using Eq. (8) one finds

$$\lambda_s^{(i)}(s) = \frac{1}{2\pi i}\frac{1}{2\,b_0}\left[\,\phi^{(i)}(a_+) - \phi^{(i)}(a_-)\,\right],\tag{26}$$

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where

$$\phi^{(2)}(a) = \frac{1}{1-a}\left[\,2 - 11\,a - 4\,(1-a)\,\ln a + 3\,(1-a)\,\ln\,(1-a)\,\right] \qquad (27)$$

and

$$\phi^{(3)}(a) = -4\,\ln a - \frac{72}{11}\frac{1}{1-a} + \frac{318}{121}\,\ln\,(1-a) + \frac{256}{363}\,\ln\,(1+\frac{9}{2}\,a)\ . \qquad (28)$$

The values of $a_\pm$ obey the following equation (from Eq. (23))

$$f(a_\pm) = f(a_0) + \frac{2\,b_0}{C}\,[\,\ln\frac{s}{Q_0^2} \pm i\,\pi\,]\ . \qquad (29)$$

In order to check our calculations we substituted $\lambda_s(s)$ obtained from (26) into Eq. (7), and found good agreement with the initial $\lambda_t^{\mathrm{eff}}(q^2)$. In Figure 3 we show the behaviour of $\mathrm{Re}\,a(s)$ and $-\,\mathrm{Im}\,a(s)$ in the s–channel, working to $O(a^3)$. For $O(a^2)$ we obtained a similar result. In Figure 4 we show the behaviour of the ratio $R(s) = \lambda_s^{(i)}(Q^2)/\lambda_t^{(i)}(q^2)$ with $s = Q^2 = -q^2$ versus s for $i = 2$ (dashed line) and $i = 3$ (solid line). The value of $R(s)$ for $s = M_\tau^2$ is about 0.93, which represents a smaller correction than the perturbative π^2-correction of Eq. (10). The agreement between successive orders is confirmation of the mechanism of induced convergence. As follows from the results we have obtained, the effects of analytic continuation can be essential for the definition of the QCD coupling constant from timelike processes, for instance from τ–decay. We plan to discuss this problem in future publications.

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<center>**Figure Captions**</center>

Figure 1 The integration contour in Eq. (4).

Figure 2 The behaviour of our running expansion parameter $a(Q^2)$ in the t–channel.

Figure 3 The behaviour of $\operatorname{Re} a(s)$ and $-\operatorname{Im} a(s)$ in the s–channel at $\O(a^3)$.

Figure 4 The behaviour of the ratio $R(s) = \lambda_s^{(i)}(Q^2)/\lambda_t^{(i)}(q^2)$ with $s = Q^2 = -q^2$
versus s for $i = 2$ (dashed line) and $i = 3$ (solid line).

<center>11</center>
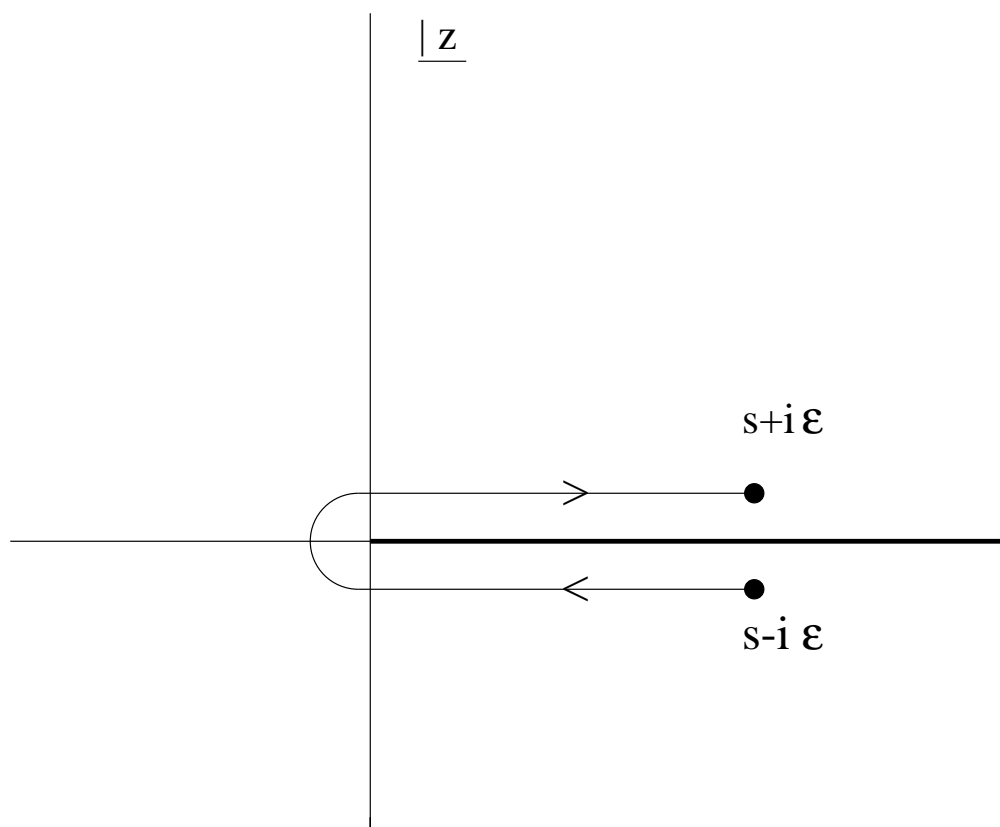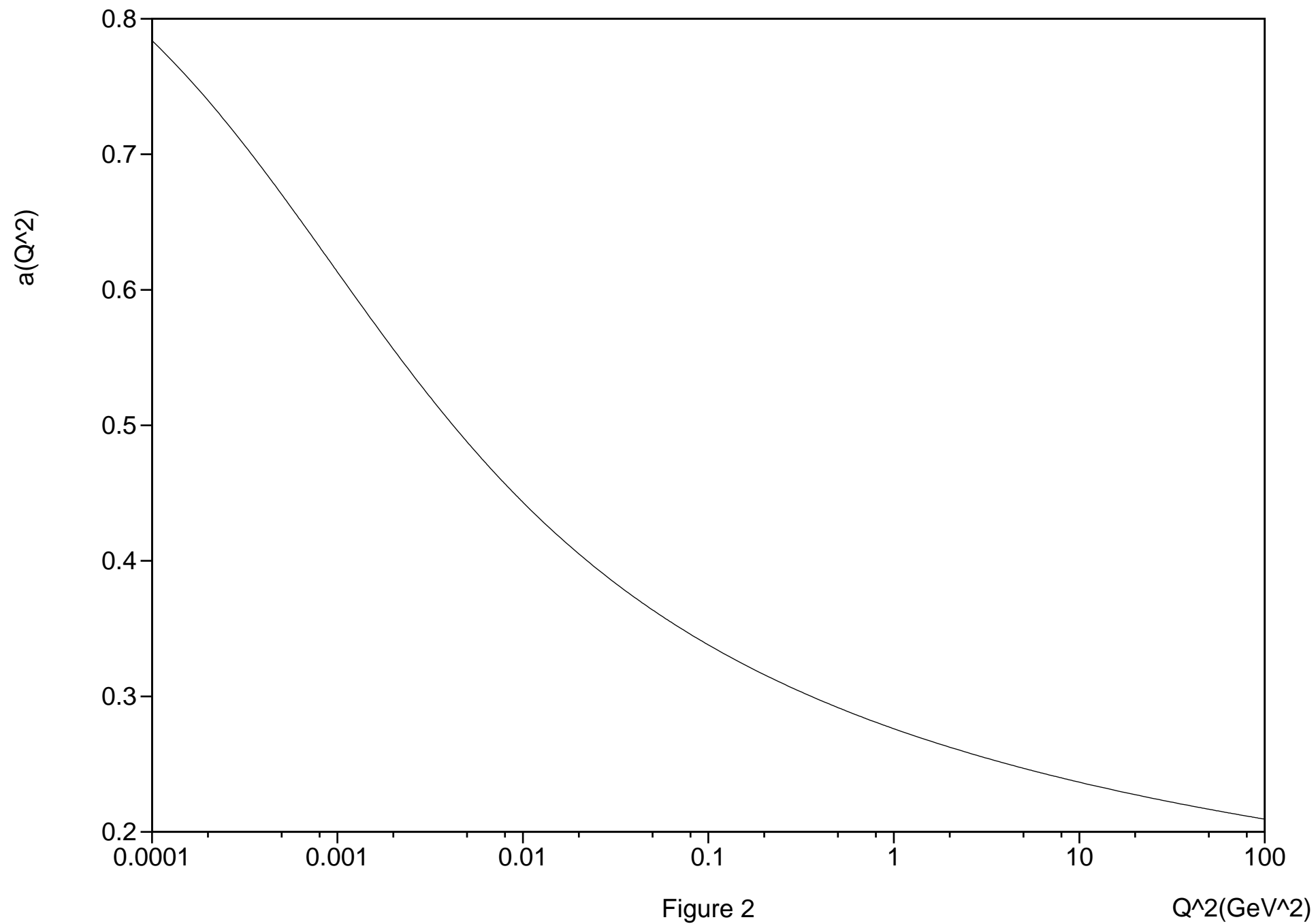
This figure "fig1-1.png" is available in "png" format from:

http://arXiv.org/ps/hep-ph/9501344v2

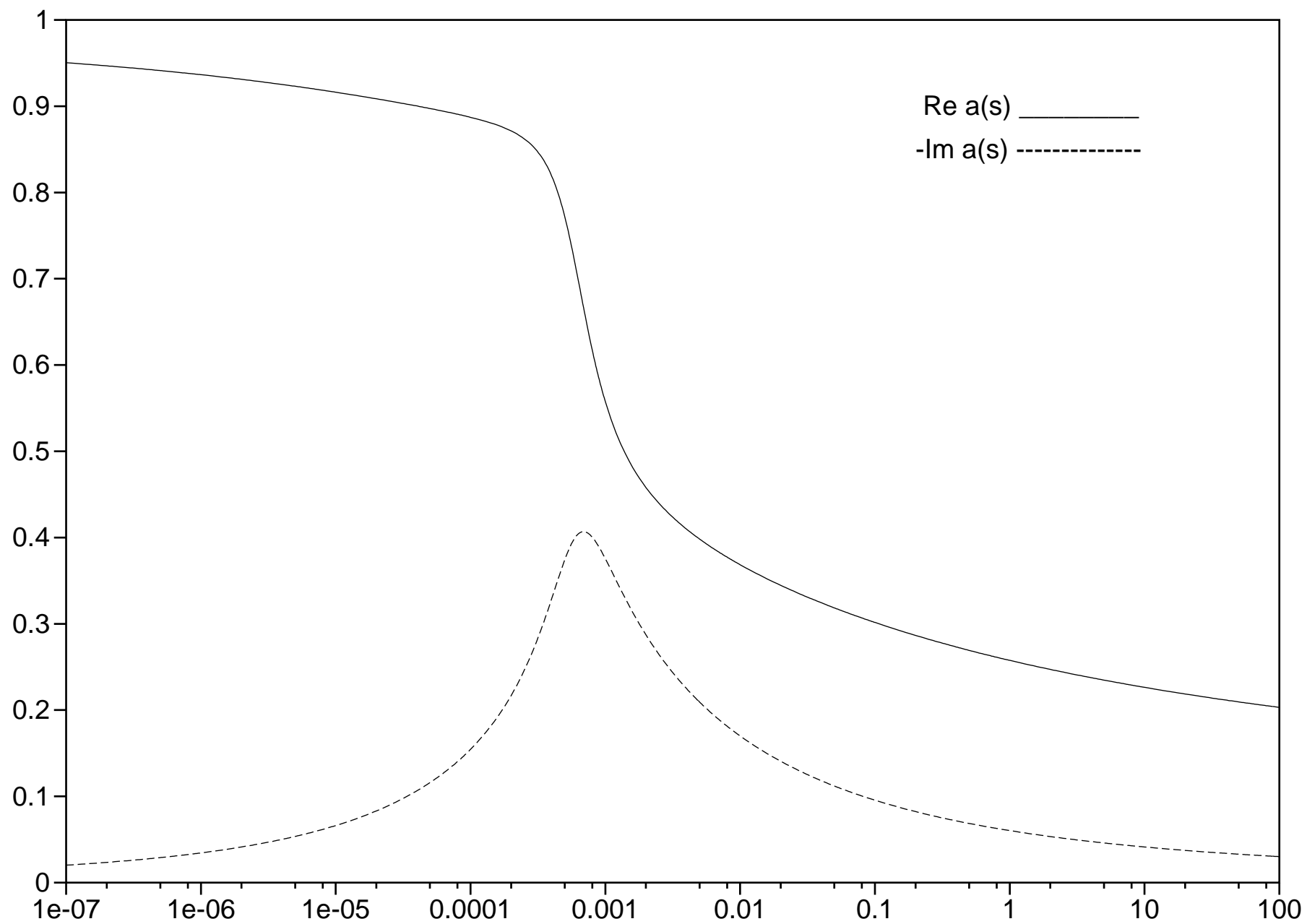Figure 1

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http://arXiv.org/ps/hep-ph/9501344v2

Figure 2

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http://arXiv.org/ps/hep-ph/9501344v2

Figure 3

Re a(s) ——————

-Im a(s) - - - - - - - -

s(GeV^2)

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Figure 4